Index theory for linear self-adjoint operator equations and nontrivial solutions for asymptotically linear operator equations

Yujun Dong*

Department of Mathematics, Nanjing Normal University, Nanjing,

Jiangsu 210097, P. R. China

E-mail: yjdong@eyou.com

Abstract

We will first establish an index theory for linear self-adjoint operator equations. And then with the help of this index theory we will discuss existence and multiplicity of solutions for asymptotically linear operator equations by making use of the dual variational methods and Morse theory. Finally, some interesting examples concerning second order Hamiltonian systems, first order Hamiltonian systems and elliptical partial differential equations will be presented to illustrate our results.

Key Words: Linear self-adjoint operator equations; index theory; relative Morse index, Ekeland type of index theory; asymptotically linear operator equations; multiple solutions; dual variational method; Morse theory; second order Hamiltonian systems; first order Hamiltonian systems; elliptical partial differential equations

.

1 Introduction

Let X be an infinite-dimensional separable Hilbert space with inner product $(\cdot,\cdot)$ and norm $||\cdot||$. Let $Y\subset X$ be a Banach space with norm $||\cdot||_Y$, and the embedding $Y\hookrightarrow X$ is compact. Let $A:Y\to X$ be continuous, selfadjoint, i.e. $(Ax,y)=(x,Ay)$ for any $x,y\in Y$, $\Im(A)$ is a closed

*Partially supported by the National Natural Science Foundation of China

subspace of X and, $\Im(A) \oplus \ker(A) = X$. In this paper for any $B \in \mathcal{L}_s(X)$ we first discuss the classification theory for

$$Ax + Bx = 0 \tag{1.1}$$

and then discuss solvability of

$$Ax + \Phi'(x) = 0 \tag{1.2}$$

where $\Phi : X \to \mathbf{R}$ is differentiable. The main results are as follows.

Definition 1.1 For any $B \in \mathcal{L}_s(X)$, we define

$$\nu_A(B) = \dim \ker(A + B).$$

$\nu_A(B)$ is called nullity of B. It will be proved in Lemma 2.1 next section that the nullity $\nu_A(B)$ is finite.

Definition 1.2 For any $B_1, B_2 \in \mathcal{L}_s(X)$ with $B_1 < B_2$, we define

$$I_A(B_1, B_2) = \sum_{\lambda \in [0,1)} \nu_A((1-\lambda)B_1 + \lambda B_2);$$

and for any $B_1, B_2 \in \mathcal{L}_s(X)$ we define

$$I_A(B_1, B_2) = I_A(B_1, k\mathrm{id}) - I_A(B_2, k\mathrm{id})$$

where $\mathrm{id} : X \to X$ is the identity map and k id$> B_1, k\mathrm{id} > B_2$ for some real number $k > 0$.

Let $B_0 \in \mathcal{L}_s(X)$ be fixed and let $i_A(B_0)$ be a prescribed integer associated with B_0.

Definition 1.3 For any $B \in \mathcal{L}_s(X)$ we define

$$i_A(B) = i_A(B_0) + I_A(B_0, B).$$

We call $i_A(B)$ index of B and $i_A(B_0)$ is called initial index. Generally, the index $i_A(B)$ depends also on B_0 and the initial index. For some well-known precise operators, we can give the initial index a special value, so that the index becomes natural. This will be done in the subsequent sections. The following proposition is also concerned with a precise example.

Proposition 1.4 If there exists $B_0 \in \mathcal{L}_s(X)$ such that $\sum_{\lambda<0} \nu_A(B_0 + \lambda id) < +\infty$, we will choose this integer for $i_A(B_0)$. Then the index defined by Definition 1.3 satisfies

$$i_A(B) = \sum_{\lambda<0} \nu_A(B + \lambda id).$$

For index and nullity defined before we have the following properties.

Proposition 1.5 (i)For any $B, B_1, B_2 \in \mathcal{L}_s(X)$, $I_A(B_1, B_2)$ and $i_A(B)$ are well-defined and finite;

(ii) For any $B_1, B_2, B_3 \in \mathcal{L}_s(X)$, $I_A(B_1, B_2) + I_A(B_2, B_3) = I_A(B_1, B_3)$;

(iii) for any $B_1, B_2 \in \mathcal{L}_s(X)$, $I_A(B_1, B_2) = i_A(B_2) - i_A(B_1)$;

(iv) for any $B_1, B_2 \in \mathcal{L}_s(X)$ with $B_1 < B_2$, $\nu_A(B_1) + i_A(B_1) \leq i_A(B_2)$.

The following theorems are concerned with solvability of equation (1.2).

Theorem 1.6 Assume that $B : X \to \mathcal{L}_s(X)$ satisfies

(1) $\Phi'(x) - B(x)x$ is bounded;

(2) there exist $B_1, B_2 \in \mathcal{L}_s(X)$ satisfying $i_A(B_1) = i_A(B_2), \nu_A(B_2) = 0$ and for any $x \in X$

$$B_1 \leq B(x) \leq B_2.$$

Then (1.2) has at least one solution.

Theorem 1.7 Assume that $\Phi''(x)$ is continuous and bounded, $\Phi'(\theta) = \theta$, and

(1) there exist $B_1, B_2 \in \mathcal{L}_s(X)$ satisfying $i_A(B_1) = i_A(B_2), \nu_A(B_2) = 0$ such that

$$B_1 \leq \Phi''(x) \leq B_2$$

for any $x \in X$ with $||x|| \geq r > 0$;

(2) with $B_0 := \Phi''(\theta)$, we have

$$i_A(B_1) \notin [i_A(B_0), i_A(B_0) + \nu_A(B_0)].$$

Then (1.2) has at least one nontrivial solution x_0. Moreover, under the further assumption that

(3)$0 = \nu_A(B_0)$ and $|i_A(B_1) - i_A(B_0)| \geq \nu_A(\Phi''(x_0))$,

equation (1.2) has at least two nontrivial solutions.

Theorem 1.8 Assume that

(1) $\Phi \in C^2(X, \mathbf{R})$ and there exist $B_1, B_2 \in \mathcal{L}_s(X)$ with $\nu_A(B_1) = 0$ such that

$$B_1 \leq \Phi''(x) \leq B_2 \forall x \in X;$$

(2)there exists $B_3 \in \mathcal{L}_s(X)$ with $B_1 < B_3$ and $i_A(B_1) = i_A(B_3), \nu_A(B_3) = 0$ such that

$$\Phi(x) \leq \frac{1}{2}(B_3 x, x) + c \forall x \in X;$$

(3) $\Phi'(\theta) = \theta, \Phi''(\theta) > B_1, \nu_A(\Phi''(\theta)) = 0$ and $i_A(\Phi''(\theta)) > i_A(B_1)$.

Then (1.2) has two distinct nontrivial solutions.

Theorem 1.9 Assume that

(1) there exist $B_1, B_2 \in \mathcal{L}_s(X)$ satisfying $B_1 \le B_2$ and $i_A(B_1) + \nu_A(B_1) = i_A(B_2), \nu_A(B_2) = 0$ such that $\Phi(x) - \frac{1}{2}(B_1 x, x)$ is convex and

$$\Phi(x) \le \frac{1}{2}(B_2 x, x) + c \forall x \in X.$$

Then (1.2) has a solution.

Moreover, if we further assume that

(2) $\Phi'(\theta) = \theta, \Phi(\theta) = 0$ and there exists $B_0 \in \mathcal{L}_s(X)$ satisfying $B_0 \ge B_1$ and

$$i_A(B_0) > i_A(B_1) + \nu_A(B_1).$$

Then (1.2) has at least one nontrivial solution.

In 1980, H. Amann and E. Zehnder[1] discussed equation (1.2) under the assumption that A :dom$(A) \subseteq X \to X$ is a unbounded selfadjoint operator. By the saddle point reduction methods they obtained some existence results for nontrivial solutions. They also discussed semilinear elliptic boundary value problems, periodic solutions of a semilinear wave equations, and periodic solutions of Hamiltonian systems as special cases of the abstract equation. In 1981, K. C. chang[2] extended their results by a simpler and unified approach. Especially, Chang obtained an existence result for three distinct solutions. Theorem 1.8 is motivated by his this result. And theorem 1.7 comes from his another result in his excellent book[3]. Chang [4]also discussed equation (1.2) by assuming that $A \in \mathcal{L}_s(X)$ and Φ' is compact. This framework can be used to discuss elliptic partial differential equations. In 1990, I Ekeland[5] discussed solvability of equation (1.2) by the dual variational methods and convex analysis theory. He assumed that $A : X \to X^*$ is closed and seladjoint. As applications he mainly focussed on second order and first order Hamiltonian systems satisfying various boundary conditions. Our theorem 1.9 generalizes his results. Some other special equations were also mentioned in the end of Chapter III.

We would like to stress that our equation (1.2) with the assumptions on the operator A supplies a new framework for some special equations. It only requires that the operator A has finite multiplicities for every eigenvalue. Most operators listed by Ekeland[5] have this property. Although this framework can not be used for studying wave equations. However, it can be used to study seconder order Hamiltonian systems, first order Hamiltonian systems as well as elliptic equations. As one can find in sections 3,4 and 5 we will obtain some new results. One can also find that the assumptions on the operator A also make us possible to establish an index theory for equation (1.1)

by the dual variational methods first. Definitions 1.1,1.2 and 1.3, and propositions 1.4 and 1.5 are concerned with this index theory. Theorems 1.6,1,7,1,8 and 1.9 can be regarded as applications of this index theory. Just because of the usage of this index theory in the assumptions of these theorems we can get some new results. Note that some special cases have been discussed by the author in [6-8].

As far as the author knows, an index theory for convex linear Hamiltonian systems was established first by I. Ekeland[9] in 1984. By the works[10-13] of Conley, Zehnder and Long, an index theory for symplectic paths was introduced. These index theories have important applications[14-22]. One can refers to the two excellent books[5,23] for systematical treatments. In [24,25] Long and Zhu defined spectral flows for paths of linear operators and relative Morse index, and redefined Maslov index for symplectic paths. Our concept of relative Morse index comes from their papers with some modifications. Note that in definition 1.2 the relative Morse index depends only on nullities. There are also other contributions on index theory. For example, in 1994, S. E. Cappell, R. lee and E. Y. Miller[26] introduced three equivalent definitions for Maslov index. For related topics one can refer to references[27-29].

The paper is organized as follows. In section 2, we will introduce an Ekeland type of index theory and prove these results. Sections 3-5 will devote to applications in some special cases of equation (1.2). Precisely, in section 3 we will first discuss second order Hamiltonian systems. Then in section 4 we will discuss first order Hamiltonian systems. Finally , in section 5 we will discuss elliptic partial differential equations.

2 Ekeland type of index theory and proofs of main results

In his excellent book[5] Ekeland introduced an index theory for convex linear Hamiltonian systems by dual variational methods. He also mentioned that by Lasry's tricks some non-convex Hamiltonian systems could be changed into convex systems and hence could be discussed also by dual variational methods. In this section we will make use of his ideas to establish an index theory for linear system (1.1) first. And then we will prove the main results in the previous section.

Let X, Y and $A : Y \to X$ be defined as before.

Lemma 2.1. For any $B \in \mathcal{L}_s(X)$, we have that $A + B : Y \to X$ is continuous, $\ker(A + B)$ is finitely dimensional, $\Im(A + B)$ is closed and

$$X = \ker(A + B) \oplus \Im(A + B).$$

Proof Because $\Im(A) \oplus \ker(A) = X$, $Y \bigcap \Im(A)$ is also a Banach space with the norm $\|\cdot\|_Y$. So $A_0 := A|_{\Im(A)\bigcap Y} : \Im(A) \bigcap Y \to \Im(A)$ is invertible and the inverse $A_0^{-1} : \Im(A) \to \Im(A) \bigcap Y \hookrightarrow \Im(A)$ is compact and self-adjoint. By the spectral theory there is a basis $\{e_j\}$ of $\Im(A)$ and a nonzero sequence $\lambda_j \to 0$ in $\mathbf{R}$ such that:

$$(e_i, e_j) = \delta_{ij} \tag{2.1}$$

$$(A_0^{-1} e_j, u) = (\lambda_j e_j, u), \ \forall \ u \in \Im(A). \tag{2.2}$$

For any $j \in \mathbf{N}$ we also have $\dim \ker(A_0^{-1} - \lambda_j) < +\infty$ and $Ae_j = \frac{1}{\lambda_j} e_j$. Fix $k \in \mathbf{R} \setminus \{0\}$ such that $\frac{1}{\lambda_j} + k \neq 0$ for any j. Then if $u = \sum_{j=1}^{\infty} c_j e_j + e_0 \in X$ with $e_0 \in \ker(A)$, we have $(A + kid)(\sum_{j=1}^{\infty}(\frac{1}{\lambda_j} + k)^{-1} c_j e_j + \frac{1}{k} e_0) = \sum_{j=1}^{\infty} c_j e_j + e_0$, and $\Im(A + kid) = X$. And for any $B \in \mathcal{L}_s(X)$, there exists a constant k such that $ker(A + kid) = \{\theta\}$ and $B - kid > 0$. Under a new inner product defined by $(x, y)_1 := ((B - kid)^{-1} x, y)$, X is also a Hilbert space and $(B - kid)(A + kid)^{-1} : X \to X$ is compact and selfadjoint. By the spectral theory again, there exists a basis $\{\zeta_j\}$ of X and a nonzero sequence $\mu_j \to 0$ such that $(B - kid)(A + kid)^{-1} \zeta_j = \mu_j \zeta_j$. Then $(A + kid)^{-1} \zeta_j := \xi_j$ is a basis of Y. This means that $(A + B)\xi_j = (A + kid + B - kid)\xi_j = (1 + \mu_j)(A + kid)\xi_j$. So $\Im(A + B) = \{\sum_{\lambda_j \neq -1} c_j \zeta_j | \sum c_j^2 < +\infty\}$ and $\ker(A + B) = \{\sum_{\mu_j = -1} c_j (B - kid)^{-1} \zeta_j\}$ is finite dimensional. Because $(B - kid)^{-1} > 0$ and $((B - kid)^{-1} \zeta_j, \zeta_i) = 0$ for $\mu_j \neq \mu_i$, the projection of $\text{span}\{(B - kid)^{-1} \zeta_j\}_{\mu_j \neq -1}$ to $\text{span}\{\zeta_j\}_{\mu_j \neq -1}$ is $\text{span}\{\zeta_j\}_{\mu_j \neq -1}$. This completes the proof. ∎

From this lemma for given $B_0 \in \mathcal{L}_s(X)$ we know that $\Lambda := (A + B_0)|_{Im(A+B_0)} : Im(A + B_0) \bigcap Y \to Im(A + B_0)$ is invertible and the inverse $\Lambda^{-1} : Im(A + B_0) \to X$ is compact. For any $B \in \mathcal{L}_s(X)$ with $B - B_0 \geq \epsilon id$ for some constant $\epsilon > 0$ we define a bilinear form:

$$\psi_{A,B|B_0}(x, y) = (\Lambda^{-1} x, y) + ((B - B_0)^{-1} x, y). \tag{2.3}$$

Note that under the inner product $((B - B_0)^{-1} x, y)$, $Im(A + B_0)$ is a Hilbert space, and $(B - B_0)\Lambda^{-1}$ is self-adjoint and compact. So there exists a basis $\{x_j\}$ of $Im(A + B_0)$ satisfying $(\Lambda^{-1} x_j, x) = \lambda_j ((B - B_0)^{-1} x_j, x)$ for every $x \in Im(A + B_0)$, $((B - B_0)^{-1} x_i, x_j) = \delta_{ij}$ and $\lambda_j \to 0$. Therefore, for any $x = \sum c_j x_j$ satisfying $\sum_{j=1}^{\infty} c_j^2 < +\infty$, we have

$$\psi_{A,B|B_0}(x, x) = \sum_{j=1}^{\infty} (1 + \lambda_j) c_j^2. \tag{2.4}$$

Define

$$E_A^+(B|B_0) := \{\sum_{j=1}^{\infty} \xi_j e_j | \xi_j = 0 \text{ if } 1 + \lambda_j \leq 0\},$$

$$E_A^0(B|B_0) := \{\sum_{j=1}^{\infty}\xi_j e_j|\xi_j = 0 \text{ if } 1+\lambda_j \neq 0\},$$
$$E_A^+(B|B_0) := \{\sum_{j=1}^{\infty}\xi_j e_j|\xi_j = 0 \text{ if } 1+\lambda_j \geq 0\}.$$

Then $E_A^0(B|B_0)$ and $E_A^-(B|B_0)$ are finitely dimensional.

Definition 2.3 For any $B \in \mathcal{L}_s(X)$ with $B - B_0 > \epsilon id$ we define

$$i_A(B|B_0) = \dim E_A^-(B|B_0), \nu_A(B|B_0) = \dim E_A^0(B|B_0).$$

This index $(i_A(B|B_0), \nu_A(B|B_0))$ is also a kind of relative index. Different from $I_A(B_0, B)$ with fixed B_0, $i_A(B|B_0)$ is only defined for $B > B_0$. The following theorem lists some properties concerning these index theories.

Theorem 2.4 (1) For any $B > B_0$ we have

$$\nu_A(B|B_0) = \nu_A(B)$$

(2) Assume $B_2 > B_1 > B_0$, then

$$i_A(B_2|B_0) \geq \nu_A(B_1|B_0) + i_A(B_1|B_0)$$

(3) For any $B_2 > B_1 > B_0$, we have

$$i_A(B_2|B_0) - i_A(B_1|B_0) = I_A(B_1, B_2).$$

(4) For $B > B_0$, we have $i_A(B|B_0) = I_A(B_0, B) - \nu_A(B_0)$.

(5) With the norm $||u||_\pm := (\pm((\Lambda^{-1}x, x) + ((B - B_0)^{-1}x, x)))^{\frac{1}{2}}$, $E_A^\pm(B|B_0)$ are Banach spaces respectively.

(6) Assume $B_0 < B_1 < B_2$ with $i_A(B_1) = i_A(B_2)$ and $\nu_A(B_2) = 0$. Then

$$X = E_A^-(B_1|B_0) \bigoplus E_A^+(B_2|B_0)$$

Proof. (1) By definition, for any $u \in E_A^0(B|B_0), v \in Im(\Lambda)$ we have

$$q_{A,B|B_0}(u, v) = 0.$$

From lemma 2.1, there exists $\xi_u \in \ker(\Lambda)$ such that

$$\Lambda^{-1}u + (B - B_0)^{-1}u = \xi_u$$

and

$$\xi_{c_1u_1+c_2u_2} = c_1\xi_{u_1} + c_2\xi_{u_2}.$$

Set $x := \Lambda^{-1}u - \xi_u$. Then $u = \Lambda(x - \xi_u) = \Lambda x$ and

$$Ax + Bx = 0.$$

So $E_A^0(B|B_0) \cong \ker(A+B)$.

(2)We only sketch the proof here. We first prove that $i_A(B|B_0)$ is a kind of Morse index: for any subspace X_1 of $\Im(A+B_0)$ satisfying $\psi_{A,B|B_0}(x,x) < 0$ on $X_1\backslash\{\theta\}$, we have that $\dim(X_1) \le i_A(B|B_0)$. Then we check : for any $x \in E_A^-(B_1|B_0) \bigoplus E_A^0(B_1|B_0)\backslash\{\theta\}$, $\psi_{A,B_2|B_0}(x,x) < 0$. Note that in the assumption we only need suppose that $((B_2 - B_1)x, x) > 0$ for any $x \in E_A^+(B_1|B_0) \setminus \{\theta\}$ in stead of $B_2 > B_1$. So in (iv) of proposition 1.5 we can assume that $((B_2 - B_1)x, x) > 0$ for any $x \in \ker(A+B_1) \setminus \{\theta\}$ in stead of $B_2 > B_1$.

(3) Write $i(s) = i_A((B_1 + s(B_2 - B_1))|B_0), \nu(s) = \nu_A((B_1 + s(B_2 - B_1))|B_0)$ for $s \in [0,1]$. From Lemma 2.6, $0 \le i(0) \le i(1) < +\infty$ and there are only finite many $s \in [0,1]$ such that $\nu(s) \neq 0$. For $s \in [0,1]$ with $\nu(s) = 0$, $i(s)$ is continuous. And for $s \in [0,1]$ with $\nu(s) \neq 0$ we have $i(s+0) = i(s-0) = \nu(s)$.

(5) From (2.4), for any $x \in E_A^+(B|B_0)$, we have $x = \sum_{1+\lambda_j>0} c_j x_j$ with $\sum_{1+\lambda_j>0} c_j^2 < \infty$. So $||x||_+ = (\sum_{1+\lambda_j>0}(1+\lambda_j)c_j^2)^{\frac{1}{2}}$ is a norm.

(6)If $u \in E_A^-(B_1|B_0)\backslash\{\theta\}$, then $\psi_{A,B_1|B_0}(u,u) < 0$ and $\psi_{A,B_2|B_0}(u,u) \le \psi_{A,B_1|B_0}(u,u) < 0$, and $u \notin E_A^+(B_2|B_0)$. So $E_A^-(B_1|B_0) \bigcap E_A^+(B_2|B_0) = \{\theta\}$ and we need only prove $X = E_A^-(B_1|B_0) + E_A^+(B_2|B_0)$. In fact, by definition we have $X = E_A^-(B_2|B_0) \bigoplus E_A^+(B_2|B_0)$, and $i_A(B_2|B_0) = \dim E_A^-(B_2|B_0) < \infty$. Let $\{e_j\}_{j=1}^{\gamma}$ be a basis of $E_A^-(B_1|B_0)$ where $\gamma := i_A(B_1|B_0)$. We have a decomposition $e_j = e_j^- + e_j^+$ with $e_j^- \in E_A^-(B_2|B_0)$ and $e_j^+ \in E_A^+(B_2|B_0)$. If $\sum_{j=1}^{\gamma} \alpha_j e_j^- = 0$, then $\bar{x} := \sum_{j=1}^{\gamma} \alpha_j e_j = \sum_{j=1}^{\gamma} \alpha_j e_j^+ \in E_A^+(B_2|B_0)$, and $\bar{x} \in E_A^-(B_1|B_0)$. So $\bar{x} = \theta$ and $\alpha_j = 0, j = 1, 2, \cdots, \gamma$. Hence $\{e_j^-\}_{j=1}^{\gamma}$ is linear independent. Since $\dim E_A^-(B_2|B_0) = i_A(B_2|B_0) = i_A(B_1|B_0) = \gamma$, $\{e_j^-\}_{j=1}^{\gamma}$ is a basis of $E_A^-(B_2|B_0)$. If $u \in X, u = u^- + u^+$ with $u^- \in E_A^-(B_2|B_0)$ and $u^+ \in E_A^+(B_2|B_0)$, then $u^- = \sum_{j=1}^{\gamma} \beta_j e_j^-$. So $u = \sum_{j=1}^{\gamma} \beta_j e_j + (u^+ - \sum_{j=1}^{\gamma} \beta_j e_j^+) := u_1 + u_2$, and $u_1 \in E_A^-(B_1|B_0)$ and $u_2 \in E_A^+(B_2|B_0)$. ∎

Proof of Proposition 1.5. We only prove(i). For any $B_1 < B_2$, by (iii) of theorem 2.4, $I_A(B_1, B_2)$ is finite and $I_A(B_1,B_2) + I_A(B_2,B_3) = I_A(B_1,B_3)$ if we further assume $B_2 < B_3$. So if $B_1, B_2 \in \mathcal{L}_s(X)$ without any restriction, there exists $\lambda_0 < 0$ such that $\nu_A(B_0 + \lambda) = 0$ for any $\lambda \le \lambda_0$. It follows that $I_A(B_1, k\mathrm{id}) - I_A(B_2, k\mathrm{id}) = I_A(B_1, k_1\mathrm{id}) - I_A(B_2, k_1\mathrm{id})$ for any $k, k_1 \in \mathbf{R}$. So the relative Morse index and hence the index $i_A(B)$ are finite and well-defined. ∎

Proof of Proposition 1.4. From the additive property,

$$I_A(B_0+\lambda, B_0) = \sum_{\lambda_0 \le \lambda < 0} \nu_A(B_0+\lambda) = i_A(B_0).$$

So $i_A(B_0+\lambda) = 0$ if $\lambda \le \lambda_0$. For any $B \in \mathcal{L}_s(X)$ there exists $\lambda_1 < 0$ with $\lambda_1 + B < B_0 + \lambda_0$. By the monotonicity of indices we have $i_A(B+\lambda) = 0$ and $\nu_A(B+\lambda) = 0$ for $\lambda \le \lambda_1$.So $I_A(B+\lambda_1, B_0+\lambda_0) \le I_A(B_0+\lambda, B_0+\lambda_0)$=0 where $\lambda < 0$ is large enough. So $i_A(B+\lambda_1) = i_A(B_0+\lambda_0) - I_A(B+\lambda_1, B_0+\lambda_0) = 0$. There exists $\lambda_2 < 0$ such that $\nu_A(B+\lambda_2) = 0$. And hence,

$$i_A(B) = I_A(B+\lambda_1, B) + i_A(B+\lambda_1) = \sum_{\lambda_1 \le \lambda < 0} \nu_A(B+\lambda) = \sum_{\lambda < 0} \nu_A(B+\lambda).$$

∎

Proof of Theorem 1.6 Choose $k \in \mathbf{R}$ with $\nu_A(k\mathrm{id}) = 0$. From lemma 2.1, $(A+k\mathrm{id})^{-1}$ is compact. And we need only verify that solutions of the following equations are a priori bounded:

$$x - k(A+k\mathrm{id})^{-1}x + (A+k\mathrm{id})^{-1}(\lambda B_1 x + (1-\lambda)\Phi'(x)) = 0$$

If not, there exist $\{x_n\} \subset X$ with $||x_n|| \to +\infty$ and $\lambda_n \in [0,1]$ such that

$$x_n - k(A+k\mathrm{id})^{-1}x_n + (A+k\mathrm{id})^{-1}(\lambda B_1 x_n + (1-\lambda)\Phi'(x_n)) = 0.$$

Set $y_n = x_n/||x_n||$ and $h(x) = \Phi'(x) - B(x)x$. Then

$$y_n - k(A+k\mathrm{id})^{-1}y_n + (A+k\mathrm{id})^{-1}(\lambda B_1 y_n + (1-\lambda)(B(x_n)y_n + ||x_n||^{-1}h(x_n))) = 0.$$

From the bounded-ness of $B(x)$, for any $y \in X$, $B(x_n)y \rightharpoonup y_1$. We define $\bar{B}y = y_1$. Then $\bar{B} \in \mathcal{L}_s(X)$ and $B_1 \le \bar{B} \le B_2$. By the compactness of $(A+k\mathrm{id})^{-1}$ and the above equation, we have $y_n \to y_0$ and $B(x_n)(y_n - y_0) \to 0$. We also assume that $\lambda_n \to \lambda_0$. Taking the limit we have

$$Ay_0 + (\lambda_0 B_1 + (1-\lambda_0)\bar{B})y_0 = 0.$$

But $B_1 \le B_3 := \lambda_0 B_1 + (1-\lambda_0)\bar{B} \le B_2$ leads to $\nu_A(B_3) = 0$. This is a contradiction to the fact that y_0 is a nontrivial solution. ∎

In the following we will prove Theorem 1.7. To do this we need a lemma, which comes from [6, Chapter II. Theorem 5.1, 5.2 and Corollary 5.2]. Note that for any $B \in \mathcal{L}_s(X)$, $m^-(B)$ denotes the multiplicity of the negative eigenvalues of B and $m^0(B)$ denotes the multiplicity of zero eigenvalues of B.

Lemma 2.5. Assume $f \in C^2(X,R)$ satisfies the (PS) condition, $f'(\theta) = \theta$, and there is a positive integer γ such that $\gamma \notin [m^-(f''(\theta)), m^0(f''(\theta)) + m^-(f''(\theta))]$ and $H_q(X, f_a; \mathbf{R}) = \delta_{q\gamma}\mathbf{R}$ for some regular value $a < 0$. Then f has a critical point $p_0 \neq \theta$ with $C_\gamma(f, p_0) \neq 0$. Moreover, if θ is a non-degenerate critical point, and $m^0(f''(p_0)) \leq |\gamma - m^-(f''(\theta))|$, then f has another critical point $p_1 \neq p_0, \theta$.

We now begin to prove Theorem 1.7. From assumption (1) and that $\Phi''(x)$ is bounded we can choose $k_1, k \in \mathbf{R}$ such that $\nu_A(kid) = 0 = \nu_A(k_1 id), B_1(t) - kid \geq id$ and

$$k_1 id \geq N''(x) \geq id \ \forall x \tag{2.5}$$

$$B_2 - kid \geq N''(x) \geq B_1 - kid \text{ for } |x| \geq r, \tag{2.6}$$

where $N(x) = \Phi(x) - \frac{1}{2}k(x,x)$. By the (iii) of Proposition 1.5, we may assume $\nu_A(kid) = 0 = \nu_A(k_1 id)$. Let $\Lambda u := Au + ku$ and consider the functional

$$\psi(u) = \frac{1}{2}(\Lambda^{-1}u, u) + N^*(u) \ \forall u \in X. \tag{2.7}$$

We have the following proposition.

Proposition 2.6 Under the assumption (i) in theorem 1.7, the functional ψ defined by (2.7) satisfies the (PS) condition.

Proof Assume $\{u_j\} \subset X$ such that $\psi(u_j)$ is bounded and $\psi'(u_j) \to \theta$ in X. If $||u_j||_X$ is bounded, then there exists a subsequence $u_{j_k} \rightharpoonup u_0$ in X, and $\Lambda^{-1}u_{j_k} \to \Lambda^{-1}u_0$. From the following (6.9), we have $\Lambda^{-1}u_{j_k} + N^{*\prime}(u_{j_k}) = \psi'(u_{j_k})$, and $N^{*\prime}(u_{j_k}) = \psi'(u_{j_k}) - \Lambda^{-1}u_{j_k} \to -\Lambda^{-1}u_0$ in X. By the Fenchel conjugate formula and [1, Theorem II.4], $u_{j_k} = N'(\psi'(u_{j_k}) - \Lambda^{-1}u_{j_k}) \to N'(-\Lambda^{-1}u_0)$ in X and ψ satisfies the (PS) condition. So in the following we only need to show $\{u_j\}$ is bounded in X.

From $N'(\theta) = \theta$, we have $N^{*\prime}(\theta) = \theta$ and

$$(\psi'(u), v) = (\Lambda^{-1}u, v) + (N^{*\prime}(u), v) \ \forall v, u \in X. \tag{2.8}$$

Noticing that $\int_0^1 N^{*\prime\prime}(\theta u_j)d\theta u_j = N^{*\prime}(u_j)$, we have

$$\Lambda^{-1}u_j + \int_0^1 N^{*\prime\prime}(su_j)ds u_j = \psi'(u_j) \to \theta, \text{ in } X. \tag{2.9}$$

If $||u_j||_X$ is not bounded, without loss of generality we assume $||u_j||_X \to \infty$. Set $x_j = u_j/||u_j||_X$. We also assume $x_j \rightharpoonup x_0$ in X by going to subsequence if necessary. And hence $\Lambda^{-1}x_j \to \Lambda^{-1}x_0$

in X. From [5,Propositions II.2.10, I.1.15] and (2.5) we have $N^{*\prime\prime}(u^*) = (N''(u))^{-1}$ as $u^* = N'(u)$, and

$$id \leq N^{*\prime\prime}(x) \leq k_1^{-1} id \ \forall\ x \in X \tag{2.10}$$

$$(B_2 - kid)^{-1} \leq N^{*\prime\prime}(x) \leq (B_1 - kid)^{-1} \ \forall\ x \in X \text{ with } ||x|| \geq r_1. \tag{2.11}$$

For any $\delta \in (0, 1)$ fixed, set

$$\begin{aligned} C_j \quad &= \int_0^1 N^{*\prime\prime}(su_j)ds, ||u_j|| \geq \frac{r_1}{\delta} \\ &= (B_1 - kid)^{-1}, \text{ otherwise} \end{aligned}$$

and

$$\xi_j = \int_0^1 N^{*\prime\prime}(su_j)dsu_j - C_j u_j.$$

From assumption(1) and (2.5)(2.9), there exists a constant $c_1 > 0$ such that

$$||\xi_j|| \leq c_1, \tag{2.12}$$

$$(1 - \delta)(B_2 - kid)^{-1} + \delta id \leq C_j \leq (1 - \delta)(B_1 - kid)^{-1} + k_1^{-1} \delta id \tag{2.13}$$

$$\Lambda^{-1} u_j + C_j u_j + \xi_j = \psi'(u_j). \tag{2.14}$$

Now by going to subsequences if necessary we may further assume $C_j u \rightharpoonup C_0 u$ in X for every $u \in X$. And from (2.9)(2.13)(2.14), for every $\epsilon > 0$ we have

$$\begin{aligned} &(B_2 - (k + \epsilon)id)^{-1} \leq C_0 \leq (B_1 - (k - \epsilon)id)^{-1}, \\ &\Lambda^{-1} x_0 + C_0 x_0 = 0 \end{aligned}$$

Let $\Lambda^{-1} x_0 = y_0$ and $B_0 = C_0^{-1} + kid$. We have

$$Ay_0 + B_0 y_0 = 0. \tag{2.15}$$

We need only show that this is a contradiction. From assumption(1) and the finiteness of the relative Morse index, for $\epsilon > 0$ is small enough, we have $\nu_A(B_1 - \epsilon id) = \nu_A(B_2 + \epsilon id) = 0$ and $i_A(B_1 - \epsilon id) = i_A(B_2 + \epsilon id)$. So that $B_1 - \epsilon id \leq B_0 \leq B_2 + \epsilon id$ and $\nu_A(B_0) = 0$. This is impossible since $||y_0||_X = 1$ and y_0 is a nontrivial solution of (2.15). This contradiction means $||u_j||_X$ is bounded. ∎

Proof of Theorem 1.7. From Lemma 2.5 and Proposition 2.6 it suffices to show that

$$H_q(X, \psi_{-a}; \mathbf{R}) \cong \delta_{q\gamma} \mathbf{R}, q = 0, 1, 2, \cdots, \tag{2.16}$$

for $a > 0$ is large enough, where $\gamma := i_A(B_1|kid)$. In fact $N^{*\prime\prime}(\theta) = (N''(\theta))^{-1}$ and

$$\begin{aligned}(\psi''(\theta)u, u) &= (\Lambda^{-1}u, u) + (N^{*\prime\prime}(\theta)u, u)\\ &= (\Lambda^{-1}u), u) + ((B_0 - kid)^{-1}u, u),\ \forall u \in X.\end{aligned}$$

By definition, $m^-(\psi''(\theta)) = i_A(B_0|kid), m^0(\psi''(\theta)) = \nu_A(B_0|kid)$. $i_A(B_1) \notin [i_A(B_0), i_A(B_0) + \nu_A(B_0)]$ if and only if $i_A(B_1|kid) \in [i_A(B_0|kid), i_A(B_0) - i|kidB) + \nu_A(B_0|kid)]$; and $\nu_A(B_0) = \nu_A(B_0|kid), |i_A(B_1) - i_A(B_0)| = |i_A(B_1|kid) - i_A(B_0|kid)|$.

We will prove (2.16) in the following two steps.

Step 1. For $\epsilon > 0$ is sufficiently small, set $\mathcal{M}_R := (E_A^+(B_2+\epsilon id|kid) \bigcap B_R) \bigoplus E_A^-(B_1-\epsilon id|kid)$, then for $R, a > 0$ are large enough we have

$$H_q(X, \psi_{-a}; \mathbf{R}) = H_q(\mathcal{M}_R, \mathcal{M}_R \cap \psi_{-a}; \mathbf{R}), q = 0, 1, 2, \cdots. \tag{2.17}$$

In fact, for any $\epsilon > 0$ is small enough we have $i_A(B_1 - \epsilon id) = i_A(B_2 + \epsilon id)$ and $\nu_A(B_2 + \epsilon id) = 0$. It is easy to see $E_A^-(B_1 - \epsilon id|kid)$ and $E_A^+(B_2 + \epsilon id|kid)$ are Banach spaces under the following norms

$$||u||_1 := ((\Lambda^{-1}u, u) + ((B_1 - kid - \epsilon id)^{-1}u, u))^{\frac{1}{2}}.$$

and

$$||u||_2 := ((\Lambda^{-1}u, u) + ((B_2(t) - kid + \epsilon id)^{-1}u, u))^{\frac{1}{2}}$$

respectively. So for every $u = u_1+u_2 \in X$ with $u_1 \in E_A^-(B_1-\epsilon id|kid)$ and $u_2 \in E_A^+(B_2+\epsilon id|kidB)$, from (2.11) we have

$$\begin{aligned}(\psi'(u), u_2 - u_1) &= (\Lambda^{-1}u, u_2 - u_1) + (N^{*\prime}(-u), u_1 - u_2)\\ &= -(\Lambda^{-1}u_1, u_1) + (\int_0^1 N^{*\prime\prime}(-\theta u)d\theta u_1, u_1)\\ &+(\Lambda^{-1}u_2, u_2) - (\int_0^1 N^{*\prime\prime}(\theta u)d\theta u_2, u_2)\\ &\geq -(\Lambda^{-1}u_1, u_1) + ((B_1 - kid - \varepsilon id)^{-1}u_1, u_1)\\ &+(\Lambda^{-1}u_2, u_2) + ((\varepsilon id + B_2(t) - kid)^{-1}u_2, u_2) - c_2\\ &\geq c_3||u_2||_X^2 + c_4||u_1||_X^2 - c_2,\end{aligned}$$

where $c_2, c_3, c_4 > 0$ are constants. When R is large enough we have

$$(\psi'(u), u_2 - u_1) > 1$$

for every $u = u_1 + u_2$ with $u_1 \in E_P^-(B_1 - \epsilon I_{2n}|B), u_2 \in E_P^+(B_2 + \epsilon I_{2n}|B)$ and $||u_2||_{L^2} \geq R$, or $||u_1||_{L^2} \geq R$. For any $u = u_2 + u_1 \notin \mathcal{M}_R$, let $\sigma(t,u) = e^{-t}u_2 + e^t u_1, T_u = \ln ||u_2|| - \ln R$, and

$$\begin{aligned}\eta(t, u_2 + u_1) \qquad &= u_2 + u_1, ||u_2|| \leq R,\\ &= \sigma(T_u t, u), ||u_2|| > R.\end{aligned}$$

Then $\eta : [0,1] \times L^2 \to L^2$ is continuous and $(\mathcal{M}_R, \mathcal{M}_R \bigcap \psi_{-a})$ is a deformation retract of (L^2, ψ_{-a}). Therefore, (2.17) is satisfied.

Step 2. For $R, a > 0$ are large enough, we have

$$H_q(\mathcal{M}_R, \mathcal{M}_R \bigcap \psi_{-a}; \mathbf{R}) \cong \delta_{q\gamma}\mathbf{R}, q = 0, 1, \cdots.$$

In fact, we have from (2.10)(2.11) that

$$\begin{aligned}&N^*(u)\\ &= (\int_0^1 \theta d\theta \int_0^1 N^{*\prime\prime}(\theta s u) ds u, u) + N^*(\theta)\\ &= \int_\delta^1 \theta d\theta \int_\delta^1 N^{*\prime\prime}(\theta s u) ds u, u) + o(1)(u,u)\\ &\leq \frac{1}{2}((B_1 - kid - \epsilon id)^{-1}u, u)\end{aligned}$$

when $||u|| \geq r_1/\delta^2$. Here in the second equality $o(1) \to 0$ as $\delta \to 0$. Hence, for every $\epsilon > 0$ there exists a constant c_5 such that

$$N^*(u) \leq (B_1 - kid - \epsilon id)^{-1}u, u) + c_5, \ \forall u \in X. \tag{2.18}$$

Therefore, for any $u = u_1 + u_2$ with $u_1 \in E_A^-(B_1 - \epsilon id|kid)$ and $u_2 \in E_A^+(B_2 + \epsilon id|kid) \bigcap B_R$, from (2.7)(2.18) we have

$$\psi(u) \leq -c_4||u_1||_X^2 + c_6||u_1||_X + c_7$$

where $c_6, c_7 > 0$. And hence,

$$\psi(u) \to -\infty \Longleftrightarrow ||u_1|| \to +\infty \text{ uniformly in } u_2 \in E_A^+(B_2 + \epsilon id|kid) \cap B_R.$$

So, there exist $T > 0, a_1 > a_2 > T, 0 < R_1 < R_2 < R_0$ such that

$$\mathcal{N}_{R_2} \subset \psi_{-a_1} \cap \mathcal{M}_{R_0} \subset \mathcal{N}_{R_1} \subset \psi_{a_2} \cap \mathcal{M}_{R_0}, \tag{2.19}$$

where $\mathcal{N}_R := (E_A^+(B_2 + \epsilon id|kid) \cap B_{R_0}) \oplus (E_P^-(B_1 - \epsilon id|kid) \setminus B_R)$ and B_R denotes the closed neighborhood of the origin with radius R in a Banach space. For any $u \in \mathcal{M}_{R_0} \cap (\psi_{-a_2} \setminus \psi_{-a_1})$,

since $\sigma(t,u) = e^{-t}u_2 + e^t u_1, \psi(\sigma(t,x))$ is continuous with respect to $t, \psi(\sigma(0,x)) = \psi(u) > -a_1$ and $\psi(\sigma(t,u)) \to -\infty$ as $t \to +\infty$, so there exists a unique $t = T_1(u)$ such that $\psi(\sigma(t,u)) = -a_1$. Because

$$\begin{aligned}&\frac{d}{dt}\psi(\sigma(t,u)) =< \psi'(\sigma(t,u)), \sigma'(t,u) >\\ &=< \psi'(e^{-t}u_2 + e^t u_1), -e^{-t}u_2 + e^t u_1 >\leq -1\end{aligned}$$

as $t > 0$, by the implicit function theorem, $t = T_1(u)$ is continuous. Define

$$\begin{aligned}\eta_1(t,u) \qquad &= u, u \in \psi_{-a_1} \cap \mathcal{M}_{R_0}\\ &= \sigma(T_1(u)t, u), u \in \mathcal{M} \cap (\psi_{-a_2} \setminus \psi_{-a_1})\end{aligned}$$

then $\eta_1 : [0,1] \times \psi_{-a_2} \cap \mathcal{M}_{R_0} \to \psi_{-a_2} \cap \mathcal{M}_{R_0}$ and $(\psi_{-a_1} \cap \mathcal{M}_{R_0}, \psi_{-a_1} \cap \mathcal{M}_{R_0})$ is a deformation retract of $(\psi_{-a_2} \cap \mathcal{M}_{R_0}, \psi_{-a_1} \cap \mathcal{M}_{R_0})$. Hence,

$$H_q(\psi_{-a_2} \cap \mathcal{M}_{R_0}, \psi_{-a_1} \cap \mathcal{M}_{R_0}) \cong H_q(\psi_{-a_1} \cap \mathcal{M}_{R_0}, \psi_{-a_1} \cap \mathcal{M}_{R_0}) \cong 0. \qquad (2.20)$$

Recall that for any topological spaces $Z \subseteq Y \subseteq X$, we have exact sequences

$$H_q(Y,Z) \to H_q(X,Z) \to H_q(X,Y) \to H_{q-1}(Y,Z).$$

From (2.19), in order to prove

$$H_q(\mathcal{M}_{R_0}, \mathcal{M}_{R_0} \cap \psi_{-a_2}) \cong H_q(\mathcal{M}, \mathcal{N}_{R_2}) \qquad (2.21)$$

we only need to prove

$$H_q(\mathcal{M}_{R_0} \cap \psi_{-a_2}, \mathcal{N}_{R_2}) \cong 0.$$

And from (2.20), it suffices to show

$$H_q(\mathcal{M}_{R_0} \cap \psi_{-a_1}, \mathcal{N}_{R_2}) \cong 0. \qquad (2.22)$$

Let $\eta : [0,1] \times \mathcal{N}_{R_1} \to \mathcal{N}_{R_1}$ satisfy

$$\begin{aligned}\eta(t, u^+ + u^-) \qquad &= u^+ + u^- \text{ if } ||u^-|| \geq R_2\\ &= u^+ + \frac{u^-}{||u^-||}(tR_2 + (1-t)||u^-||) \text{ if } R_2 > ||u^-|| \geq R_1.\end{aligned}$$

Set $\tau_1 = \eta_1(1,\cdot)$ and $\xi = \tau_1 \circ \eta : [0,1] \times (\psi_{-a_1} \cap \mathcal{M}) \to \psi_{-a_1} \cap \mathcal{M}$. Then $(\mathcal{N}_{R_2}, \mathcal{N}_{R_2})$ is a deformation retract of $(\psi_{-a_1} \cap \mathcal{M}_{R_0}, \mathcal{N}_{R_2})$. As a result, (2.22) and (2.21) are valid. From (2.21) we have

$$\begin{aligned}&H(\mathcal{M}_{R_0}, \mathcal{M}_{R_0} \cap \psi_{-a_2}; \mathbf{R})\\ &\cong H_q(\mathcal{M}_{R_0}, \mathcal{N}_{R_2}; \mathbf{R})\\ &\cong H_q(E_P^-(B_1 - \epsilon I_{2n}|B) \cap B_{R_1}, \partial(E_P^-(B_1 - \epsilon I_{2n}|B) \cap B_{R_1}); \mathbf{R})\\ &\cong \delta_{q\gamma}\mathbf{R}, q = 0, 1, 2 \cdots.\end{aligned}$$

∎

Proof of Theorem 1.8. Define $\Lambda x = Ax+B_1 x, N(x) = \Phi(x)+\frac{1}{2}(B_1 x, x), N^*(x) = \sup_{y\in X}\{(x,y)-N(y)\}$ and

$$\psi(u) = \frac{1}{2}(\Lambda^{-1}u, u) + N^*(u) \quad u \in X. \tag{2.23}$$

From assumption (2), we have

$$\psi(u) \geq \frac{1}{2}[(\Lambda^{-1}u, u) + ((\bar{B}_2 - B_1)^{-1}u, u)] - c \quad \forall u \in X. \tag{2.24}$$

So ψ is bounded from below. If $\psi(u_j)$ is bounded, we have $||u_j||_X$ is also bounded. We can assume $u_j \rightharpoonup u_0$ and $\Lambda^{-1}u_j \to \Lambda^{-1}u_0$. If $\psi'(u_j) \to 0$, we have $N^*(u_j) = \psi'(u_j) - \Lambda^{-1}u_j \to -\Lambda^{-1}u_0$ and $u_j = N'(\psi'(u_j) - \Lambda^{-1}u_j) \to N'(-\Lambda^{-1}u_0)$ in X. So ψ satisfies the (PS) condition. It is easy to check that $\psi''(\theta) : X \to X$ is invertible; $m^-(\psi''(\theta)) = i_A(B_0|B_1) > 0$, so that θ is not a minimal point. From a theorem in [4],this complete the proof. ∎

Proof of Theorem 1.9. Consider the functional defined in (2.23). At this time its domain is not X but $Im(\Lambda)$. We also have inequality (2.24) with $\bar{B}_2$ and X instead with B_2 and $Im(\Lambda)$respectively. ψ is bounded from below. Let $u_n \in Im(\Lambda)$ satisfying $\psi(u_n) \to Inf\psi > -\infty$. Then $\{u_n\}$ is bounded and we assume that $u_n \rightharpoonup u_0$ in $Im(\Lambda)$. By the compactness of Λ_0^{-1} and the weakly lower semi-continuity of N^*, we have $Inf\psi(u) \geq \psi(u_0)$. This means that u_0 is a critical point of ψ. A simple calculation shows that $(\Lambda_0^{-1}u_0 + N^*(u_0), u) = 0$ for any $u \in Im(\Lambda)$. So $x = \Lambda_0^{-1} + x_0$ is a solution of (1.2) for some $x_0 \in \ker(\Lambda)$. When $\Phi'(\theta) = \theta$, then θ is a solution of (1.2). We will prove that $u_0 \neq \theta$ under assumption (2). In fact, we have

$$\psi(u) \leq \frac{1}{2}[(\Lambda_0^{-1}u, u) + ((B_0 - B_1)^{-1}u, u)] \text{ as } u \to \theta.$$

The Morse index of the right functional at $u = \theta$ is $\dim(E_A^-(B_0|B_1))$. So for any $u \in E_A^-(B_0|B_1)\backslash\{\theta\}$ small enough, we have $\psi(u) < 0 = \psi(\theta)$. Hence $u_0 \neq \theta$ and $x = \Lambda_0^{-1} + x_0 \neq \theta$. This completes the proof. ∎

3 Second order Hamiltonian systems

In this section we will make use of proposition 1.4 to give some classifications for second order Hamiltonian systems.

3.1 Sturm-Liouville BVPs

In this subsection we will establish a classification theory for the following Lagrangian system satisfying Sturm-Liouville BVPs

$$(\Lambda(t)x')' + B(t)x = 0 \tag{3.1}$$

$$x(0)\cos\alpha - \Lambda(0)x'(0)\sin\alpha = 0 \tag{3.2}$$

$$x(1)\cos\beta - \Lambda(1)x'(1)\sin\beta = 0 \tag{3.3}$$

where $\Lambda \in C([0,1]; GL_s(\mathbf{R}^n)), B \in L^\infty((0,1); GL_s(\mathbf{R}))$ is positive definite for $t \in [0,1]$, $0 \leq \alpha < \pi$ and $0 < \beta \leq \pi$. Let $X := L^2((0,1); GL_s(\mathbf{R}^n)), Y = \{x \in C^1([0,1], \mathbf{R}^n) | (\Lambda(t)x'(t))' \in L^2((0,1); \mathbf{R}^n), x(t)$ satisfies (3.2)(3.3)$\}$. For every $x \in Y$, we define $||x||_Y := (\int_0^1 |x(t)|^2 + |x'(t)| + |(\Lambda(t)x'(t))'|^2 dt)^{\frac{1}{2}}$, $A : X \to Y$ by $(Ax)(t) := (\Lambda(t)x'(t))'$. Then X is a separable Hilbert space, Y is a Banach space and the embedding $Y \hookrightarrow X$ is compact. Define $(Bx)(t) = B(t)x(t)$ for any $x \in X$. Then equation (3.1)(3.2)(3.3) is equivalent to equation (1.1). In view of the following lemma 3.1, the following problem (3.2)(3.3) and

$$(\Lambda(t)x')' - (\bar\lambda + 1)x = 0$$

has no nontrivial solutions. From general theory of ordinary differential equations(c.f. [30, pp407-408] for example), for any $h \in L^2((0,1); \mathbf{R}^n)$, the following problem (3.2)(3.3) and

$$(\Lambda(t)x')' - (\bar\lambda + 1)x = h(t)$$

has a unique solution. So from lemma 2.1, A is continuous and closed and, $\ker(A) \bigoplus \Im(A) = X$.

Lemma 3.1 There exists $\bar\lambda > 0$ such that

$$\int_0^1 (\Lambda(t)x'(t))' \cdot x(t) dt \leq \bar\lambda \int_0^1 |x(t)|^2 dt \forall x \in Y.$$

Proof.As $\alpha = 0$, we have $x(0) = 0$; as $\alpha \neq 0$, we have $\Lambda(0)x'(0) = x(0)\cot\alpha$. By partial integration we have

$$\int_0^1 (\Lambda(t)x'(t))') \cdot x(t) dt = -\int_0^1 \Lambda(t)x'(t) \cdot x'(t) dt + \Lambda(1)x'(1) \cdot x(1) - \Lambda(0)x'(0) \cdot x(0).$$

So we need only prove that: for any given $a > 0$, there exists $\lambda_a > 0$ such that

$$\int_0^1 |x'(t)|^2 dt + \lambda_a \int_0^1 |x(t)|^2 dt \geq a(|x(0)|^2 + |x(1)|^2).$$

The following trick comes from Professor Eric Sere:

$$\frac{d}{dt}|x(t)|^2 = \frac{1}{2}x'(t) \cdot x(t) \geq -\epsilon|x'(t)|^2 - \frac{1}{\epsilon}|x(t)|^2,$$
$$|x(t)|^2 \geq -\epsilon \int_0^1 |x(t)|^2 dt - \frac{1}{\epsilon}\int_0^1 |x(t)|^2 dt + |x(0)|^2,$$
$$(1 + \frac{1}{\epsilon})\int_0^1 |x(t)|^2 dt + \epsilon \int_0^1 |x'(t)|^2 dt \geq |x(0)|^2$$

where $\epsilon > 0$ is a constant. This completes the proof. ∎

Lemma 3.1 shows that $\ker(A - \lambda I_n) = \{\theta\}$ for $\lambda > \bar{\lambda}$. In view of proposition 1.4 we can give the following definition.

Definition 3.2 For any $B \in L^\infty((0,1); GL_s(\mathbf{R}^n))$, we define

$$\nu^s_{\Lambda,\alpha,\beta}(B) := \dim\ker(A+B),$$
$$i^s_{\Lambda,\alpha,\beta}(B) := \sum_{\lambda<0} \nu^s_{\Lambda,\alpha,\beta}(B + I_n\lambda).$$

For any $B_1, B_2 \in L^\infty((0,1);\mathrm{GL}_s(\mathbf{R}^n))$, we define $B_1 \le B_2$ if and only if $B_1(t) \le B_2(t)$ for a.e. $t \in (0,1)$; and define $B_1 < B_2$ if and only if $B_1 \le B_2$ and $B_1(t) < B_2(t)$ on a subset of $(0,1)$ with positive measure.

Proposition 3.3 We have the following property:

(1) For any $B \in L^\infty((0,1); GL_s(\mathbf{R}^n))$, we have

$$\nu^s_{\Lambda,\alpha,\beta}(B) \in \{0, 1, \cdots, n\}.$$

(2) For any $B_1, B_2 \in L^\infty((0,1);\mathrm{GL}_s(\mathbf{R}^n))$ satisfying $B_1 < B_2$, we have $i^s_{\Lambda,\alpha,\beta}(B_1) + \nu^s_{\Lambda,\alpha,\beta}(B_1) \le i^s_{\Lambda,\alpha,\beta}(B_2)$.

Proof (1)Let $y(t) = \Lambda(t)x'(t), z = (y, x)$, then (3.1)-(3.3) has an equivalent form:

$$\dot{z} = J\mathrm{diag}\{\Lambda(t)^{-1}, B(t)\}z, \tag{3.4}$$
$$x(0)\cos\alpha - y(0)\sin\alpha = 0,$$
$$x(1)\cos\beta - y(1)\sin\beta = 0.$$

Let $\gamma(t)$ be the fundamental solution of (3.4). Then

$$\begin{aligned}
\ker(A+B) &= \{z(t) = \gamma(t)c | c \in \mathbf{R}^{2n}, z = (y,x)\text{satisfies}(3.2)(3.3)\} \\
&\cong \{c_1, c_2 \in \mathbf{R}^n | c_1\cos\alpha - c_2\sin\alpha = 0, (I_n\cos\beta, -I_n\sin\beta)\gamma(1)(c_1,c_2)^\tau = 0\} \\
&\cong \{c \in \mathbf{R}^n | (I_n\cos\beta, -I_n\sin\beta)\gamma(1)(0,c)^\tau = 0\} \subseteq \mathbf{R}^n,
\end{aligned}$$

as $\alpha = 0$.

(2) Follows directly from the (iii) of proposition 1.5 and the proof of (2) of theorem 2.4. ∎

We now begin to discuss solvability of the following nonlinear Hamiltonian systems:

$$(\Lambda(t)x')' + V'(t,x) = 0, \tag{3.5}$$
$$x(0)\cos\alpha - \Lambda(0)x'(0)\sin\alpha = 0$$
$$x(1)\cos\beta - \Lambda(1)x'(1)\sin\beta = 0$$

where $V : [0,1] \times \mathbf{R}^n \to \mathbf{R}^n$ is continuous and $V'(t,x)$ denotes the gradient of $V(t,x)$ with respect to x.

Define $\Phi(x) = \int_0^1 V(t,x(t))dt$ for every $x \in X$. Then $\Phi'(x) = V'(\cdot, x(\cdot))$ when $V \in C^1([0,1] \times \mathbf{R}^n, \mathbf{R})$, and $\Phi''(x) = V''(\cdot, x(\cdot))$ when $V \in C^2([0,1] \times \mathbf{R}^n, \mathbf{R})$. Obviously, equation (3.5)(3.2)(3.3) is equivalent to equation (1.2). From theorem 1.6, theorem 1.7 and its proof we have the following results.

Theorem 3.4 Assume that $V \in C^1([0,1]\times\mathbf{R}^n, \mathbf{R})$ and there exist $B_1, B_2 \in L^\infty((0,1);\mathrm{GL}_s(\mathbf{R}^n))$ with $B_1 \le B_2, i^s_{\Lambda,\alpha,\beta}(B_1) = i^s_{\Lambda,\alpha,\beta}(B_2)$, $\nu^s_{\Lambda,\alpha,\beta}(B_2) = 0$ and $B \in C([0,1] \times \mathbf{R}^n, GL_s(\mathbf{R}^n))$ such that

$$V'(t,x) - B(t,x)x \text{ is bounded}$$

$$B_1(t) \le B(t,x) \le B_2(t), (t,x) \in (0,1) \times \mathbf{R}^n, \text{ with } |x| \ge r > 0.$$

Then (3.1)(3.2)(3.3) has at least one solution.

Theorem 3.5 Assume

(1) $V \in C^2([0,1] \times \mathbf{R}^n, \mathbf{R}), B_1(t) \le V''(t,x) \le B_2(t)$ for $|x| \ge r > 0$ with $i^s_{\Lambda,\alpha,\beta}(B_1) = i^s_{\Lambda,\alpha,\beta}(B_2), \nu^s_{\Lambda,\alpha,\beta}(B_2) = 0$.

(2) $V'(t,0) \equiv 0, \bar{B}(t) := V''(t,0)$ and $i^s_{\Lambda,\alpha,\beta}(B_1) \notin [i^s_{\Lambda,\alpha,\beta}(\bar{B}), i^s_{\Lambda,\alpha,\beta}(\bar{B}) + \nu^s_{\Lambda,\alpha,\beta}(\bar{B})$.

Then problem (3.1)(3.2)(3.3) has at least one nontrivial solution. Moreover, if we further assume

(3)$\nu^s_{\Lambda,\alpha,\beta}(\bar{B}) = 0, |i^s_{\Lambda,\alpha,\beta}(B_1) - i^s_{\Lambda,\alpha,\beta}(\bar{B})| \ge n$.

Then (3.1)(3.2)(3.3) has two nontrivial solutions.

Remarks 1. As $\alpha = 0, \beta = \pi, \Lambda(t) \equiv I_n$, linear system (3.1)(3.2)(3.3) reduces to

$$x'' + B(t)x = 0$$

$$x(0) = 0 = x(1).$$

An index theory $(i(B), \nu(B))$ was established in [6](2005) by making use a direct variational method. Note that this index theory is a special case of definition 3.2, i.e., $(i(B), \nu(B)) = (i^s_{I_n,0,\pi}(B), \nu^s_{I_n,0,\pi}(B))$. The index theory $(i(B), \nu(B))$ was used to discuss associated second order nonlinear Hamiltonian systems. Note that most of the main results in [6] are covered by theorems 3.4, 3.5. For related topics one can refers to [31-33].

2. As $n = 1, \Lambda(t) = 1$, equation (3.5) is called Duffing equation as usual and can be expressed as

$$x'' + f(t,x) = 0.$$

Many papers devoted to solvability of this Duffing equation satisfying various boundary conditions(see [34-42] and the references therein). One can find that the main results of some of these papers are special cases of theorem 3.4 or the following theorem 3.10.

3.2 Generalized periodic boundary value problems

Consider the following problem (3.1)(3.6)

$$\begin{aligned} &(\Lambda(t)x')' + B(t)x = 0 \\ &x(1) = Mx(0), x'(1) = Nx'(0) \end{aligned} \tag{3.6}$$

where $M \in GL(n), M^\tau \Lambda(1)N = \Lambda(0), \Lambda \in C([0,1]; GL_s(n))$ and $\Lambda(t)$ is positive definite. Define $X := L^2((0,1); \mathbf{R}^n), Y := \{x : [0,1] \to \mathbf{R}^n | (\Lambda(t)x'(t))' \in L^2(0,1; \mathbf{R}^n)$ and x satisfies (3.6)$\}$. Then $Y \hookrightarrow X$ is compact. Define $(Ax)(t) := (\Lambda(t)x'(t))'$ for every $x \in Y$. Then $A : Y \to X$ is continuous and $X = \ker(A) \bigoplus \Im(A)$. A simple calculation shows that

$$\int_0^1 [(\Lambda(t)x'(t))' \cdot x(t)]dt \leq 0 \tag{3.7}$$

for every $x \in Y$. So similar to definition 3.2 we have from proposition 1.4 the following definition.

Definition 3.6 For any $B \in L^\infty(0,1; GL_s(\mathbf{R}^n))$ we define

$$\begin{aligned} &\nu^s_{\Lambda,M}(B) := \dim Ker(A+B), \\ &i^s_{\Lambda,M}(B) := \sum_{\lambda<0} \nu^s_{\Lambda,M}(B + I_n\lambda). \end{aligned}$$

Proposition 3.7. (1)For any $B \in L^\infty((0,1);\mathrm{GL}_s(\mathbf{R}^n))$, we have that $E^0(\Lambda, B, M)$ is the solution subspace of (3.1)(3.6) and $\nu^s_{\Lambda,M}(B) \in \{0,1,2,\cdots,2n\}$.

(2) For any $B_1, B_2 \in L^\infty((0,1);\mathrm{GL}_s(\mathbf{R}^n))$, if $B_1 \leq B_2$, we have $i^s_{\Lambda,M}(B_1) \leq i^s_{\Lambda,M}(B_2)$; if $B_1 < B_2$, we have $i^s_{\Lambda,M}(B_1) + \nu^s_{\Lambda,M}(B_1) \leq i^s_{\Lambda,M}(B_2)$.

(3)For any Λ_1, Λ_2 with $\Lambda_1(1) = \Lambda_1(0), \Lambda_2(1) = \Lambda_2(0)$, if $\Lambda_1 < \Lambda_2$, then $i^s_{\Lambda_1,M}(B) + \nu^s_{\Lambda_1,M}(B) \leq i^s_{\Lambda_2,M}(B)$.

(4)If $B_i \in L^\infty((0,1);\mathrm{GL}_s(\mathbf{R}^{n_i})), \Lambda_i \in C([0,1];\mathrm{GL}_s(\mathbf{R}^{n_i})), M_i, N_i \in \mathrm{GL}(\mathbf{R}^{n_i})$ with $M_i^T\Lambda_i(1)N_i = \Lambda_i(0), i = 1,2$ and $B = \mathrm{diag}\{B_1, B_2\}, \Lambda = \mathrm{diag}\{\Lambda_1, \Lambda_2\}, M = \mathrm{diag}\{M_1, M_2\}, N = \mathrm{diag}\{N_1, N_2\}$ then $i^s_{\Lambda,M}(B) = i^s_{\Lambda_1,M_1}(B_1) + i^s_{\Lambda_2,M_2}(B_2), \nu^s_{\Lambda,M}(B) = \nu^s_{\Lambda_1,M_1}(B_1) + \nu^s_{\Lambda_2,M_2}(B_2)$.

Example 3.8. Let $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$ be the eigenvalues of a constant matrix A. Then

$$\begin{aligned} &i^s_{\lambda I_n, I_n}(A) = {}^\#\{k : \alpha_k > 0\} + 2\sum_{k=1}^n {}^\#\{j \in \mathbf{N} : 4\lambda j^2\pi^2 < \alpha_k\}, \\ &\nu^s_{\lambda I_n, I_n}(A) = {}^\#\{k : \alpha_k = 0\} + 2\sum_{k=1}^n {}^\#\{j \in \mathbf{N} : 4\lambda j^2\pi^2 = \alpha_k\}, \end{aligned}$$

$$i^s_{\lambda I_n,-I_n}(A)=2\sum_{k=1}^{n}{}^{\#}\{j\in\mathbf{N}:\lambda(2j-1)^2\pi^2<\alpha_k\},$$
$$\nu^s_{\lambda I_n,-I_n}(A)=2\sum_{k=1}^{n}{}^{\#}\{j\in\mathbf{N}:\lambda(2j-1)^2\pi^2=\alpha_k\}.$$

where ${}^{\#}S$ denotes the number of elements in a set S. For $a\in\mathbf{R}\setminus\{\pm1,0\}$, we have with $\mu_0=\arccos\frac{2}{a^{-1}+a}$ that

$$i^s_{\lambda I_n,aI_n}(A)=\sum_{1}^{k}{}^{\#}\{j\in\mathbf{N}:\lambda(2j\pi+\mu_0)^2<\alpha_k\}+\sum_{k=1}^{n}{}^{\#}\{j\in\mathbf{N}:\lambda(2\pi-\mu_0+2j\pi)^2<\alpha_k\},$$
$$\mu^s_{\lambda I_n,aI_n}(A)=\sum_{1}^{k}{}^{\#}\{j\in\mathbf{N}:\lambda(2j\pi+\mu_0)^2=\alpha_k\}+\sum_{k=1}^{n}{}^{\#}\{j\in\mathbf{N}:\lambda(2\pi-\mu_0+2j\pi)^2=\alpha_k\}.$$

Remark 3.9. The first two formulae in Example 3.8 were given first by Mawhin and Willem in the book[43] when $\lambda=1$. In order to discuss minimal periodic solution problems Y. Long[44,45] established two kind of index theory for linear Hamiltonian systems satisfying periodic boundary value conditions in some sense of symmetries in 1993 and 1994.

We discuss solvability of the following nonlinear systems (3.8)(3.6):

$$(\Lambda(t)x')'+B(t,x)x+h(t,x)=0, \tag{3.8}$$
$$x(1)=Mx(0),x'(1)=Nx'(0)$$

where $A:\in C([0,1]\times\mathbf{R}^n,GL_s(\mathbf{R}^n)),h:[0,1]\times\mathbf{R}^n\to\mathbf{R}^n$ are continuous. Generally, (3.7)(3.6) is not a Lagrangian system, i.e., we can not find a $V\in C^1([0,1]\times\mathbf{R}^n,\mathbf{R})$ such that $V'(t,x)=+A(t,x)x+h(t,x)$. Even though we still have the following theorem, which proof is similar to theorem 1.6's.

Theorem 3.10 Assume

(1) there exist $B_1,B_2\in L^\infty((0,1);\mathrm{GL}_s(\mathbf{R}^n))$ with $B_1\le B_2, i^s_{\Lambda,M}(B_1)=i^s_{\Lambda,M}(B_2)$, $\nu^s_{\Lambda,M}(B_2)=0$ such that

$$B_1(t)\le B(t,x)\le B_2(t),x\in R^n,\ \text{a.e. } t\in(0,1);$$

(2) $h(t,x)=(|x|)$ as $|x|\to+\infty$. Then (3.8)(3.6) has at least one solution.

Example 3.11 Let $B(t,x)=B_1(t)\cos^2|x|^2+B_2(t)\sin^2|x|^2,h(t,x)=x(1+|x|^2)\sin|x|t$. As $\Lambda(t)=I_n,M=N=-I_n$, choose $B_1(t)=(\pi^2(2k-1)^2+\epsilon)I_n,B_2(t)=(\pi^2(2k+1)^2-\epsilon)I_n$; as $\Lambda(t)=I_n,M=N=I_n$, choose $B_1(t)=(4\pi^2k^2+\epsilon)I_n,B_2(t)=(4\pi^2(k+1)^2-\epsilon)I_n$; as

$\Lambda(t) = I_n, M = \lambda I_n, N = (\lambda)^{-1} I_n$ with $\lambda \in \mathbf{R} \setminus \{\pm 1, 0\}$, choose $B_1(t) = ((2k\pi + \mu)^2 + \epsilon) I_n, B_2(t) = ((2k\pi + 2\pi - \mu)^2 - \epsilon) I_n$ with $\mu = \arccos \frac{2}{\lambda^{-1} + \lambda}$. Then (3.8)(3.6) has at least one solution provided $\epsilon > 0$ is sufficiently small.

Finally, we will consider the following Lagrangian system (3.5)(3.6)

$$(\Lambda(t)x')' + V'(t,x) = 0,$$
$$x(1) = M_1 x(0), x'(1) = M_2 x'(0)$$

From theorem 1.7 and its proof we have the following theorem.

Theorem 3.12 Assume

(1) $V \in C^2([0,1] \times \mathbf{R}^n, \mathbf{R}), B_1(t) \le V''(t,x) \le B_2(t)$ for $|x| \ge r > 0$ with $i^s_{\Lambda,M}(B_1) = i^s_{\Lambda,M}(B_2), \nu^s_{\Lambda,M}(B_2) = 0$.

(2) $V'(t,0) \equiv 0, \bar{B}(t) := V''(t,0)$ and $i^s_{\Lambda,M}(B_1) \notin [i^s_{\Lambda,M}(\bar{B}), i^s_{\Lambda,M_1}(\bar{B}) + \nu^s_{\Lambda,M_1}(\bar{B})]$.

Then problem (3.5)(3.6) has at least one nontrivial solution. Moreover, if we assume

(3)$\nu^s_{\Lambda,M}(\bar{B}) = 0, |i^s_{\Lambda,M}(\bar{B}) - i^s_{\Lambda,M}(\bar{B})| \ge 2n$.

Then (3.5)(3.6) has two nontrivial solutions.

Remark 3.13. I. Ekeland[5 suggested to discuss the following boundary value conditions

$$\begin{pmatrix} x(1) \\ x'(1) \end{pmatrix} = M \begin{pmatrix} x(0) \\ x'(0) \end{pmatrix}, \tag{3.9}$$

where $M \in GL(2n)$ satisfying

$$M^T \begin{pmatrix} 0 & -\Lambda(1) \\ \Lambda(1) & 0 \end{pmatrix} M = \begin{pmatrix} 0 & -\Lambda(0) \\ \Lambda(0) & 0 \end{pmatrix}$$

Condition (3.6) is a special case of (3.9). This condition is chosen because we can get an inequality like (3.7), and so we can establish an index theory like in definition 3.6. In next section we will discuss generalized periodic boundary condition for first order Hamiltonian system, which will cover condition (3.9).

4 First order Hamiltonian systems

4.1 Bolza BVPs

In this subsection we will establish a classification theory for the following Hamiltonian system

$$\dot{x} = JB(t)x \tag{4.1}$$

$$x_1(0)\cos\alpha + x_2(0)\sin\alpha = 0 \tag{4.2}$$

$$x_1(1)\cos\beta + x_2(1)\sin\beta = 0 \tag{4.3}$$

where $B \in L^\infty((0,1); GL_s(\mathbf{R}^{2n})), 0 \le \alpha < \pi$ and $0 < \beta \le \pi, x = (x_1, x_2) \in \mathbf{R}^n \times \mathbf{R}^n$. Let $X := L^2((0,1); \mathbf{R}^{2n}), Y = \{x : [0,1] \to \mathbf{R}^{2n} | x' \in L^2((0,1); \mathbf{R}^{2n}), x(t)$ satisfies (4.2)(4.3)$\}$. Define $A : Y \to X$ by $(Ax)(t) := Jx'(t)$. We can choose suitable value $\lambda \in \mathbf{R}$ such that problem (4.1)(4.2)(4.3) with $B(t)$ replaced by λI_{2n} has no nontrivial solutions. From general theory of ordinary differential equations, for any $h \in L^2((0,1), \mathbf{R}^{2n})$ the following problem (4.2)(4.3) and

$$J\dot{x} + \lambda x = h(t)$$

has a unique solution. So from lemma 2.1 A is continuous and closed and, $\ker(A) \bigoplus \Im(A) = X$.

Definition 4.1 For any $B \in L^\infty((0,1); GL_s(\mathbf{R}^{2n}))$, we define

$$\begin{aligned}
&\nu^f_{\alpha,\beta}(B) := \dim\ker(A+B),\\
&i^f_{\alpha,\beta}(\mathrm{diag}\{0, I_n\}) := i^s_{I_n,\alpha,\beta}(0),\\
&i^f_{\alpha,\beta}(B) := i^f_{\alpha,\beta}(\mathrm{diag}\{0, I_n\}) + I^f_{\alpha,\beta}(\mathrm{diag}\{0, I_n\}, B);
\end{aligned}$$

and

$$\begin{aligned}
&I^f_{\alpha,\beta}(B_1, B_2) = \sum_{\lambda\in[0,1)} \nu^f_{\alpha,\beta}((1-\lambda)B_1 + \lambda B_2) \quad \text{as} \quad B_1 < B_2,\\
&I^f_{\alpha,\beta}(B_1, B_2) = I^f_{\alpha,\beta}(B_1, kid) - I^f_{\alpha,\beta}(B_2, kid) \quad \text{for every} \quad B_1, B_2 \quad \text{with} \quad kid > B_1, kid > B_2.
\end{aligned}$$

Proposition 4.2 We have the following property:

(1) For any $B \in L^\infty((0,1); GL_s(\mathbf{R}^{2n}))$, we have

$$\nu^f_{\alpha,\beta}(B) \in \{0, 1, \cdots, n\}.$$

(2) For any $B_1, B_2 \in L^\infty((0,1); \mathrm{GL}_s(\mathbf{R}^{2n}))$ satisfying $B_1 < B_2$, we have $i^f_{\alpha,\beta}(B_1) + \nu^f_{\alpha,\beta}(B_1) \le i^f_{\alpha,\beta}(B_2)$.

(3) For any $B \in L^\infty((0,1); \mathrm{GL}_s(\mathbf{R}^n))$, we have

$$(i^f_{\alpha,\beta}(\mathrm{diag}\{B, I_n\}), \nu^f_{\alpha,\beta}(\mathrm{diag}\{B, I_n\})) = (i^s_{I_n,\alpha,\beta}(B), \nu^s_{I_n,\alpha,\beta}(B)).$$

Proof. We only prove

$$i^f_{\alpha,\beta}(\mathrm{diag}\{B, I_n\}) = i^s_{I_n,\alpha,\beta}(B). \tag{4.4}$$

Case 1: $B > 0$. Choose a negative number $c \in \mathbf{R}$. Similar to the (iii) of theorem 2.4, we have

$$\begin{aligned}
&i^f_{\alpha,\beta}(\mathrm{diag}\{B, I_n\}) - i^f_{\alpha,\beta}(\mathrm{diag}\{0, I_n\})\\
&= i^f_{\alpha,\beta}(\mathrm{diag}\{B, I_n\}|cI_{2n}) - i^f_{\alpha,\beta}(\mathrm{diag}\{0, I_n\}|cI_{2n})
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\lambda\in[0,1)} \nu^f_{\alpha,\beta}(\mathrm{diag}\{\lambda B, I_n\}) \\
&= \sum_{\lambda\in[0,1)} \nu^s_{I_n,\alpha,\beta}(\lambda B) \\
&= i^s_{I_n,\alpha,\beta}(B) - i^s_{I_n,\alpha,\beta}(0)
\end{aligned}
$$

Combining the second formula in definition 4.1, formula(4.4) follows.

Case 2: B is arbitrary. Choose a positive number c such that $cI_n > B$. Similar to Case 1, we have

$$
\begin{aligned}
&i^f_{\alpha,\beta}(\mathrm{diag}\{c_1 I_n, I_n\}) - i^f_{\alpha,\beta}(\mathrm{diag}\{B, I_n\}) \\
&= i^s_{I_n,\alpha,\beta}(c_1 I_n) - i^s_{I_n,\alpha,\beta}(B)
\end{aligned}
$$

and hence formula (4.4). ▮

We now begin to discuss solvability of the following nonlinear Hamiltonian systems:

$$
\begin{aligned}
&x'' = JH'(t,x), \qquad (4.5)\\
&x_1(0)\cos\alpha + x_2(0)\sin\alpha = 0\\
&x_1(1)\cos\beta + x_2(1)\sin\beta = 0
\end{aligned}
$$

where $H : [0,1]\times \mathbf{R}^{2n} \to \mathbf{R}^{2n}$ is differentiable and $H'(t,x)$ is the gradient of H with respect to x.

From theorems 1.6 1nd 1.7 we have the following theorems.

Theorem 4.3 Assume

(1) there exist $B_1, B_2 \in L^\infty((0,1);\mathrm{GL}_s(\mathbf{R}^{2n}))$ with $B_1 \le B_2, i^f_{\alpha,\beta}(B_1) = i^f_{\alpha,\beta}(B_2)$, $\nu^f_{\alpha,\beta}(B_2) = 0$ such that

(2) $H'(t,x) - B(t,x)$ is bounded, $B : [0,1]\times\mathbf{R}^{2n} \to GL_s(\mathbf{R}^{2n})$ is continuous and

$$B_1(t) \le B(t,x) \le B_2(t), (t,x) \in (0,1)\times\mathbf{R}^{2n}, \text{ with } |x| \ge r > 0.$$

Then (4.4)(4.2)(4.3) has at least one solution.

Theorem 4.4 Assume

(1) $H \in C^2([0,1]\times\mathbf{R}^{2n},\mathbf{R}), B_1(t) \le H''(t,x) \le B_2(t)$ for $|x| \ge r > 0$ with $i^f_{\alpha,\beta}(B_1) = i^f_{\alpha,\beta}(B_2), \nu^f_{\alpha,\beta}(B_2) = 0$.

(2) $H'(t,0) \equiv 0, \bar{B}(t) := H''(t,0)$ and $i^f_{\alpha,\beta}(B_1) \notin [i^f_{\alpha,\beta}(\bar{B}), i^f_{\alpha,\beta}(\bar{B}) + \nu^f_{\alpha,\beta}(\bar{B})]$.

Then problem (4.4)(4.2)(4.3) has at least one nontrivial solution. Moreover, if we further assume

(3)$\nu^f_{\alpha,\beta}(\bar{B}) = 0, |i^f_{\alpha,\beta}(B_1) - i^f_{\alpha,\beta}(\bar{B})| \ge n$.

Then (4.4)(4.2)(4.3) has two nontrivial solutions.

Note that in [7] we discussed the special case $\alpha = 0, \beta = \pi$.

4.2 Generalized periodic boundary value problems

Consider the following problem (4.1)(4.6)

$$x' = JB(t)x$$
$$x(1) = Px(0) \tag{4.6}$$

where $P \in Sp(2n)$ is prescribed. Define $X := L^2((0,1);\mathbf{R}^{2n}), Y := \{x : [0,1] \to \mathbf{R}^{2n} | x' \in L^2(0,1;\mathbf{R}^{2n})$ and x satisfies (4.6)$\}$. Then $Y \hookrightarrow X$ is compact. Define $(Ax)(t) := Jx'(t)$ for every $x \in Y$. Similar to Proposition 7 in page 22 of Ekeland's book[5], for the given $P \in Sp(2n)$ there exists $\lambda \in \mathbf{R}$ such that $e^{J\lambda} - P$ is invertible. So (4.1)(4.6) with $B(t)$ replaced by λI_{2n} has only the trivial solution. Thus, from lemma 2.1 $A : Y \to X$ is continuous, closed and $X = \ker(A) \bigoplus \Im(A)$.

Choose $i_P^f(0) := i_P(I_{2n})$ defined by definition 2.2 in [8]. we have the following definition.

Definition 4.5 For any $B \in L^\infty((0,1);\mathrm{GL}_s(\mathbf{R}^{2n}))$, we define

$$\nu_P^f(B) = dim \ker(A+B),$$
$$i_P^f(B) = i_P^f(0) + I_P^f(0,B);$$

and

$$I_P^f(B_1,B_2) = \sum_{\lambda \in [0,1)} \nu_P^f((1-\lambda)B_1 + \lambda B_2) \quad \text{as} \quad B_1 < B_2,$$
$$I_P^f(B_1,B_2) = I_P^f(B_1,kid) - I_P^f(B_2,kid) \quad \text{for every} \quad B_1, B_2 \quad \text{with} \quad kid > B_1, kid > B_2.$$

From theorem 1.5 we have the following proposition.

Proposition 4.6. (1)For any $B \in L^\infty((0,1);\mathrm{GL}_s(\mathbf{R}^{2n}))$, we have $\nu_P^f(B) \in \{0,1,2,\cdots,2n\}$.

(2) For any $B_1, B_2 \in L^\infty((0,1);\mathrm{GL}_s(\mathbf{R}^{2n}))$ satisfying $B_1 < B_2$, we have $i_P^f(B_1) + \nu_P^f(B_1) \leq i_P^f(B_2)$.

We now discuss solvability of the following nonlinear system (4.6)(4.5):

$$x' = JH'(t,x)$$
$$x(1) = Px(0)$$

where $H : [0,1] \times \mathbf{R}^{2n} \to \mathbf{R}$ is differentiable and $P \in Sp(2n)$ is prescribed.

Similar to theorems 4.3 and 4.4 we have

Theorem 4.7 Assume

(1) there exist $B_1, B_2 \in L^\infty((0,1);\mathrm{GL}_s(\mathbf{R}^{2n}))$ with $B_1 \le B_2, i_P^f(B_1) = i_P^f(B_2)$, $\nu_P^f(B_2) = 0$ such that

$$B_1(t) \le B(t,x) \le B_2(t), x \in R^n, \text{ a.e. } t \in (0,1);$$

(2) $H'(t,x) - B(t,x)$ is bounded.

Then (4.4)(4.5) has at least one solution.

Theorem 4.8 Assume

(1) $H \in C^2([0,1] \times \mathbf{R}^{2n}, \mathbf{R}), B_1(t) \le H''(t,x) \le B_2(t)$ for $|x| \ge M > 0$ with $i_P^f(B_1) = i_P^f(B_2), \nu_P^f(B_2) = 0$.

(2) $H'(t,0) \equiv 0, \bar{B}(t) := H''(t,0)$ and $i_P^f(B_1) \notin [i_P^f(\bar{B}), i_P^f(\bar{B}) + \nu_P^f(\bar{B})]$.

Then problem (4.4)(4.5) has at least one nontrivial solution. Moreover, if we assume

(3)$\nu_P^f(\bar{B}) = 0, |i_P^f(B_1) - i_P^f(\bar{B})| \ge 2n$.

Then (4.4)(4.5) has two nontrivial solutions.

Note that in [8] the above problem have been discussed already separately.

5 Second order elliptic partial differential equations

In this section we will discuss index theory for linear elliptic equations satisfying Dirichlet boundary conditions and nontrivial solutions for nonlinear elliptic equations. First we consider the following linear systems:

$$\Delta u + b(x)u = 0, x \in \Omega \tag{5.1}$$

$$u|\partial\Omega = 0 \tag{5.2}$$

where $\Omega \in \mathbf{R}^n$ is a bounded open domain, and its boundary $\partial\Omega$ is smooth, $b \in L^\infty(\Omega)$.

Define $X := L^2(\Omega), Y := H_0^2(\Omega)$ and $Au = \triangle u, (Bu)(x) = b(x)u(x)$. Then the embedding $Y \hookrightarrow X$ is compact, $A : Y \to X$ and $B : X \to X$ are continuous and self-adjoint. It is well-known that the spectrum $\sigma(-A) \subset (0, +\infty)$. And by the Dirichlet Principle, for any $f \in L^2(\Omega)$ equation $\Delta u = f$ and (5.2) has a weak solution. This weak solution is also classical solution. So $Im(A) = X$ and $\ker(A) = \{\theta\}$. From proposition 1.4 we can give the following definition.

Definition 5.1 For any $b \in L^\infty(\Omega)$, we define

$$\nu_\Delta(b) = \dim\ker(A + B)$$

$$i_\Delta(b) = \sum_{\lambda<0} \nu_\Delta(b + \lambda).$$

The following proposition comes from proposition 1.5 directly. Note that for any $b_1, b_2 \in L^\infty(\Omega)$, we define $b_1 \le b_2$ if and only if $b_1(x) \le b_2(x)$ for a.e. $x \in \Omega$; and define $b_1 < b_2$ if and only if $b_1 \le b_2$ and $b_1(x) < b_2(x)$ on a subset of Ω with positive measure.

Proposition 5.2 (1) $\nu_\Delta(b)$ is finite.

(2)For any $b_1 < b_2$ belonging to $L^\infty(\Omega)$, we have

$$\nu_\Delta(b_1) + i_\Delta(b_1) \le i_\Delta(b_2).$$

Finally, we consider the following problem:

$$\begin{aligned} &\Delta u + f(x,u) = 0, x \in \Omega \\ &u|\partial\Omega = 0 \end{aligned} \tag{5.3}$$

Define $F(x,u) := \int_0^u f(x,s)ds$ and $\Phi(u) = \int_\Omega F(x,u(x))dx$. Then $\Phi'(u) = f(\cdot,u(\cdot)), \Phi''(u) = \frac{\partial}{\partial u} f(\cdot,u(\cdot))$ and equation (5.3)(5.2) is equivalent to (1.2).

Theorem 5.3 Assume that

$$b_1(x) \le f(x,u)/u \le b_2(x), |u| > r > 0$$

and $i_\Delta(b_1) = i_\Delta(b_2), \nu_\Delta(b_2) = 0$. Then (5.3)(5.2) has at least one solution.

Proof. Define $g(x,u) = f(x,u)/u$ as $|u| > r$; $g(x,u) = b_1(x)x$ as $|u| \le r$, and $B(u) = g(\cdot,u(\cdot))$. Then from theorem 1.6 and its proof we can complete the proof. ∎

And from theorems 1.7, 1.8 and 1.9 we obtain the following results.

Theorem 5.4 Assume

(1) $f \in C^1(\Omega \times \mathbf{R}, \mathbf{R}), b_1(x) \le \frac{\partial}{\partial u} f(x,u) \le b_2(x)$ for $|u| \ge r > 0$ with $i_\Delta(b_1) = i_\Delta(b_2), \nu_\Delta(b_2) = 0$.

(2) $f(x,0) \equiv 0, \bar{b}(x) := \frac{\partial}{\partial u} f(x,0)$ and $i_\Delta(b_1) \notin [i_\Delta(\bar{b}), i_\Delta(\bar{b}) + \nu_\Delta(\bar{b})]$.

Then (5.3)(5.2) has at least one nontrivial solution.

Theorem 5.5 Assume that

(1) $f' \in C(\Omega \times \mathbf{R}, \mathbf{R})$ and there exist $b_1, b_2 \in L^\infty(\Omega)$ with $\nu_\Delta(b_1) = 0$ such that

$$b_1 \le f;(x,u) \le b_2 \forall (x,u) \in \Omega \times \mathbf{R};$$

(2)there exists $b_3 \in L^\infty(\Omega)$ with $b_1 < b_3$ and $i_\Delta(b_1) = i(b_3), \nu_\Delta(b_3) = 0$ such that

$$\Phi(x) \le \frac{1}{2}(b_3(x)x, x) + c \forall x \in X;$$

(3) $f(x,0) = 0, \frac{\partial}{\partial u} f(x,0) > b_1(x), \nu_A(\frac{\partial}{\partial u} f(\cdot,0)) = 0$ and $i_\Delta(\frac{\partial}{\partial u} f(\cdot,0)) > i_\Delta(b_1)$.

Then (5.3)(5.2) has two distinct nontrivial solutions.

Theorem 5.6 Assume that

(1) there exist $b_1, b_2 \in L^\infty(\Omega)$ satisfying $b_1 \leq b_2$ and $i_\Delta(b_1) + \nu_\Delta(b_1) = i_\Delta(b_2), \nu_\Delta(b_2) = 0$ such that $\int_0^u f(x,s)ds - \frac{1}{2}(b_1(x)u^2$ is convex with respect to u and

$$\int_0^u f(x,s)ds \leq \frac{1}{2}b_2(x)u^2 + c\forall(x,u) \in \Omega \times \mathbf{R}.$$

Then (5.3)(5.2) has a solution.

Moreover, if we further assume that

(2) $f(x,0) = 0$ and there exists $b_0 \in L^\infty(\Omega)$ satisfying $b_0 \geq b_1$ and

$$i_\Delta(b_0) > i_\Delta(b_1) + \nu_\Delta(b_1).$$

Then (5.3)(5.2) has at least one nontrivial solution.

Remark 5.7 Theorems 5.4, 5.5 and 5.6 cover some results in [3, Chapter III].

Acknowledgement Part of the manuscript was finished during my stay at IHES and Universite Paris-Dauphine from Aug 2004 to Oct 2005. I would like to express my sincere thanks to Profs Jean-Pierre BOURGUIGNON and Eric Sere and other members and visitors of the two institutes for their warm helps. Special thanks are devoted to Eric Sere for offering a proof for lemma 3.1. During the preparation of the paper I also visited Chern Institute of Mathematics invited by Prof Yiming Long. I also would like to express my thanks to Prof Yiming Long and Weiping Zhang for their hospitality. The final manuscript was finished while my visiting at PIMS. I thank Prof Ivar Ekeland for his invitation and help.

References

[1] H. Amann and E. Zehnder, Nontrivial soultions for a class of nonresonance problems and applications to nonlinear differential equations, Annali Scuola Norm. Sup. Pisa 7(1980)439-603.

[2] K. C. Chang, Solutions of asymptotically linear operator equations via Morse theory, Comm. Pure Appl. Math. 34(1981)693-712.

[3] K. C. Chang, Infinite dimentional Morse theory and multiple solution problems. Birkhauser. Basel(1993).

[4] K. C. Chang, Critical point theory and its application, Shanghai Sci. Tech. Press(1986)(in Chinese).

[5] I. Ekeland, Convexity methods in Hamiltonian mechanics. Springer-Verlag. Berlin. 1990.

[6] Y. Dong, Index theory, nontrivial solutions and asymptotically linear second order Hamiltonian systems. J. Differ. Equations 214(2005)233-255.

[7] Y. Dong, Maslov type index theory for linear Hamiltonian systems with Bolza boundary value conditions and multiple solutions for nonlinear Hamiltonian systems. Pacific J Math (2005)253-280.

[8] Y. Dong, $P-$index theory for linear Hamiltonian systems and multiple solutions for nonlinear Hamiltonian systems. Nonlinearity 19(2006)1275-1294.

[9] I. Ekeland, Une theorie de Morse pour les systemes hamiltoniens convexes. Ann IHP "Analyse non lineaire" 1(1984)19-78.

[10] C. Conley and E Zehnder, Morse-type index theory for flows and periodic solutions for Hamiltonian equations. Comm. Pure Appl. Math. 37(1984)207-253.

[11] Y. Long and E. Zehnder, Morse theory for forced oscillations of asymptotically linear Hamiltonian systems. Stock. Process. Phys. Geom. ed S Alberverio et al(Teaneck, NJ:World Scientific)(1990)528-563.

[12] Y. Long, Maslov-type index, degenerate critical poin ts, and asymptotically linear Hamiltonian systems. Sci. China 33(1990)1409-1419.

[13] Y. Long, A Maslov-type index theory for symplectic p aths. Topol. Methods Nonlinear Anal. 10(1997)47-78.

[14] I. Ekeland and H. Hofer, Periodic solutions with prescribed period for convex autonomous Hamitonian systems Invent. Math. 81(1985)155-188.

[15] D. Dong and Y. Long, The iteration formula of Maslov-type index theory with applications to nonlinear Hamiltonian systems Trans. American Math. Soc. 349(1997)2619-2661.

[16] I. Ekeland and H. Hofer, Convex Hamiltonian energy surfaces and their closed trajectories. Comm. Math. Phys. 113(1987)419-467.

[17] Y. Long and C. Zhu, Closed characteristics on compact convex hypersurfaces in $\mathbf{R}^{2n}$. Ann. Math. 155(2002)317-368.

[18] C. Liu, Y. Long and C. Zhu, Multiplicity of closed characteristics on symmetric convex hypersurfaces in $\mathbf{R}^{2n}$. Math. Ann. 323(2002)201-215.

[19] G. Fei, Relative Morse index and its applications to the Hamiltonian systems in the presnece of symmetry. J. Diff. Equa. 122(1995)302-315.

[20] G. Fei, Maslov-type index and periodic solution of asymptotically linear Hamiltonian systems which are resonant at infinity, J. Differential Equations 121(1995)121-133.

[21] J. Su, Nontrivial periodic solutions for the asymptotically linear Hamiltonian systems with resonance at infinity. J. Differential Equations 145(1998)252-273.

[22] Y. Guo, Nontrivial periodic solutions for asymptotically linear Hamiltonian systems with resonance. J. Differential Equations 175(2001)71-87.

[23] Y. Long, Index theory for symplectic paths with applications, Progress in Math. No. 207, Birkhäuser. Basel. 2002.

[24] C. Zhu and Y. Long, Maslov type index theorey for symplectiuc paths and spectral flow(I). Chinese Ann. of Math. 20B(1999)413-424.

[25] Y. Long and C.Zhu, Maslov type index theorey for symplectiuc paths and spectral flow(II). Chinese Ann. of Math. 21B(2000) 89-108.

[26] S. Cappell, Lee R and Miller E Y, On the Maslov index. Comm. Pure Appl. Math. 17(1994)121-186.

[27] J. Leray, Lagrangian Analysis and quantum mechanics, a mathematical structure related to asymptotic expansions and the Maslov index(Cambridge, MA:MIT Press)1981.

[28] P. Dazord, Invariants homotopiques attachs aux fibrs symplectiques Ann. Inst. Fourier 29(1979)25-78.

[29] de Gosson M, The structure of $q-$symplectic geometry J. Math Pures Appl. 71(1992)429-453.

[30] P. Hartman, Ordinary differential equations. Second edition(1982). Birkhauser. Boston Basel Stuttgart.

[31] I. Ekeland, N. Ghoussoub and H. Tehrani, Multiple solutions for a classical problem in the calculus of variations, J. Differential Equations 131 (1996)229-243

[32] F. Clarke and E. Ekeland, Nonlinear oscillations and boundary value problems for Hamiltonian systems, Arch. Rational Mech. Anal. 78(1982)315-333.

[33] Z. Wang, Multiple solutions for infinite functional and applications to asymptotically linear problems, Math. Sinica(N.S.)5(1989)101-113.

[34] Y. Dong, On Equivalent Conditions for the Solvability of Equation $(p(t)x')' + f(t,x) = h(t)$ Satisfying Linear Boundary Conditions with f Restricted by Linear Growth Conditions, J. Math. Anal. Appl. 245 (2000)204-220.

[35] Y. Dong, On the solvability of asymptotically positively homogeneous equations with S-L boundary value conditions, Nonlinear Analysis 42(2000) 1351-1363.

[36] H. Wang and Y. Li, Existence and uniqueness of periodic solutions for Duffing equations across many points of resonance. J. Differential Equations 108 (1994)152–169

[37] H. Wang and Y. Li, Two-point boundary value problems for second order ordinary differential equations across many resonant points. J. Math. Anal. Appl. 179 (1993)61–75

[38] C. Fabry, Landesman-Lazer conditions for periodic boundary value problems with asymmetric nonlinearities. J. Differential Equations 116 (1995)405–418

[39] S. Villegas, A Neumann problem with asymmetric nonlinearity and a related minimizing problem. J. Differential Equations 145 (1998)145–155

[40] R. Iannacci, M. Nkashama, Nonlinear elliptic partial differential equations at resonance: higher eigenvalues. Nonlinear Anal. 25 (1995)455–471

[41] R. Iannacci, M. Nkashama, and J. Ward, Nonlinear second order elliptic partial differential equations at resonance. Trans. Amer. Math. Soc. 311 (1989)711–726

[42] M. Nkashama, S. Robinson, Resonance and nonresonance in terms of average values. J. Differential Equations 132 (1996)46–65

[43] J. Mawhin and M. Willem, Critical point theory and Hamiltonian systems. Springer. Berlin 1998.

[44] Y. Long, The minimal period problem for classical Hamiltonian systems with even potentials. Ann. Inst. H. Poincare Anal. non lineaire. 10(1993)605-626.

[45] Y. Long, The minimal period problem of periodic solutions for autonomous superquadratic second order Hamiltonian systems. J. Differential Equations 111(1994)147-174.